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# Analytic Determination of the Complex-Field Zeros of REM

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**Abstract**

The complex field zeros of the Random Energy Model are analytically determined. For $T < T_c$ they are distributed in the whole complex plane with a density that decays very fast with the real component of H. For $T > T_c$ a region is found which is free of zeros and is associated to the paramagnetic phase of the system. This region is separated from the former by a line of zeros. The real axis is found to be enclosed in a cloud of zeros in the whole frozen phase of the system.

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## 1. Introduction

The determination of the zeros of the partition function allows much insight into the behavior of a model. Since their relation with phase transitions were clarified by the works of Yang and Lee [1,2] and M. Fisher [3], many interesting properties of zeros distributions have been studied in homogeneous systems [4-9].

More limited, on the other hand, are the results concerning zeros distributions of disordered models. Taking into account that all the information about a given model is contained in the zeros of its partition function, it is reasonable to expect that many of the interesting properties [10] of disordered systems will be somehow reflected by them.

An example of this can be found in the complex magnetic field zeros of spin glasses, which are *dense* near the real axis. This property has been suggested by Parisi [11] as being a consequence of the multiplicity of states and was recently illustrated [17] by a numerical calculation of the complex field zeros distribution in the Random Energy Model (REM) [12].

Here an analytical calculation of the complex field zeros of REM is reported. The technique is the same as recently employed by Derrida [13] to calculate the complex $\beta$ zeros of the same model, which had been also numerically estimated [14,15].

In section 2 we describe the basics of the REM and give an account of the ideas that allow the calculation of its zeros. Section 3 is devoted to the REM in a complex field and the calculation of the dominant contribution to its partition function in each region of the complex $h = \beta H$ plane. In section 4 these results are used to obtain the density of zeros of REM in the complex field plane, while section 5 contains our conclusions.

## 2. THE RANDOM ENERGY MODEL

The REM is defined [12] to have $2^N$ independent random energy levels $E_i$, i=1,..$2^N$ distributed according to $P(E) = (\pi N)^{-1/2} \exp(-E^2/N)$. The partition function of a *sample* (a sample is a set of $2^N$ numbers $E_i$), $\mathcal{Z} = \sum_{i=1}^{2^N} e^{-\beta E_i}$ can be written as $\mathcal{Z} = \sum_E n(E)\, e^{-\beta E}$ where $n(E)$ is the number of levels between $E$ and $E + \Delta E$. This density of states $n(E)$ if found to satisfy

$$\begin{cases} \langle n(E) \rangle \sim \exp N(\log 2 - (\frac{E}{N})^2) \\ \\ \langle n(E)^2 \rangle - \langle n(E) \rangle^2 \sim \langle n(E) \rangle \end{cases} \tag{1}$$

so two different energy regions can be identified:

If $|E| < N\epsilon_c$ , with $\epsilon_c = (\log 2)^{1/2}$, then the number of states $n(E)$ is much greater than one and its fluctuations are small, so for a typical sample we will have [13]

$$n^{\text{typ}}(E) = \langle n(E) \rangle + \eta_E \langle n(E) \rangle^{1/2} \tag{2}$$

where $\eta_E$ is a random number of order 1.

On the other hand, if $|E| > N\epsilon_c$ , then the number of states is exponentially small meaning that a typical sample will have no levels in this region

$$n^{\text{typ}}(E) = 0 \tag{3}$$

The partition function for a typical sample will then be

$$\mathcal{Z}^{\text{typ}} = A + B \tag{4}$$

with

$$A = \sum_{|E|<N\epsilon_c} \langle n(E) \rangle e^{-\beta E} \tag{5}$$

and

$$B = \sum_{|E|<N\epsilon_c} \eta_E \langle n(E) \rangle^{1/2} e^{-\beta E} \tag{6}$$

The calculation of $A$ may be done by the steepest descent method [16] for complex $\beta = \beta_1 + i\beta_2$.

$$A \;\sim\; 2^N \int_{-\epsilon_c}^{+\epsilon_c} \mathrm{d}e \quad \exp -N(e^2 + \beta e) \tag{7}$$

The integration contour is stretched to infinity through paths of steepest descent for the real part of the exponent of the integrand. When the integration limits lay on opposite sides of the saddle point in $e = -\beta/2$ we can write

$$\int_{-\epsilon_c}^{+\epsilon_c} = \int_{-\epsilon_c}^{-\infty} + \int_{-\infty}^{+\infty} + \int_{+\infty}^{+\epsilon_c} \tag{8}$$

and $A$ has contributions both from the saddle point and from the integration limits for large N. This happens if $|\beta_1| < \beta_c = 2\epsilon_c = 2(\log 2)^{1/2}$ and we get

$$A \;\sim\; \exp\{N\beta\epsilon_c\} + \exp\{N(\log 2 + (\beta/2)^2)\} \tag{9}$$

whereas if $|\beta_1| > \beta_c$ the two steepest descent paths run from the integration limits down to infinity on the same side of the saddle point so it does not contribute and we have

$$\int_{-\epsilon_c}^{+\epsilon_c} = \int_{-\epsilon_c}^{\pm\infty} + \int_{\pm\infty}^{+\epsilon_c} \tag{10}$$

so

$$A \;\sim\; \exp\{N\beta\epsilon_c\} \tag{11}$$

It has to be mentioned [13] that the lowest lying states in REM have order one fluctuations from sample to sample [12] so the contributions coming from the integration limits, both in (9) and (11), are known up to a fluctuating complex (if $\beta$ is complex) factor of order one. We will return to this point later on because it results crucial for our calculation.

The contribution $B$ of the fluctuations can be calculated for large N. Owing to the randomness of $\eta_E, B$ is not the integral of an analytic function so the steepest descent method is no longer applicable. On the other hand, the $\eta_E$ are uncorrelated for different values of $E$ so the term with the largest modulus will dominate the sum in (6). We can then estimate the modulus of $B$ as

$$|B| \sim \max_{-N\epsilon_c \leq E \leq N\epsilon_c} |\langle n(E)\rangle^{1/2} e^{-\beta E}\ | \tag{12}$$

It is easily seen that when $|\beta_1| < \beta_c/2$ the maximum is in between the limits so

$$|B| \sim \exp\{\frac{N}{2}(\log 2 + \beta_1^2)\} \tag{13}$$

while if $|\beta_1| > \beta_c/2$ one of the limits dominates and

$$|B| \sim \exp\{N|\beta_1|\epsilon_c)\} \tag{14}$$

We will now use these results to calculate $\mathcal{Z}$ for a typical sample of REM when a complex field $H$ acts on it. We will take $\beta$ real from now on, as well as $h_1$ and $h_2$ non-negative ($h = \beta H = h_1 + ih_2$) which can be done without any loss of generality.

## 3. REM IN A COMPLEX FIELD

When a magnetic field $H$ acts on the system, the energies $E_i$ of the states now have, in addition to the internal energies (coming from interactions among spins) a contribution from the interaction with the field, $E_i = U_i - HM_i$, where the $M_i$ are the magnetizations of the states. The partition function $\mathcal{Z} = \sum_i\ e^{-\beta E_i}$ is then

$$\mathcal{Z}(\beta, h) = \sum_{i=1}^{2^N} \quad e^{-\beta U_i + hM_i} \tag{15}$$

The internal energies $U_i$ have a Gaussian distribution which is independent of the field, so taking into account that among the $2^N$ states of the system, $g_m = g_M = \binom{N}{\frac{(N+M)}{2}}$ have magnetization $M = Nm$, we can write

$$\mathcal{Z}(\beta, h) \;=\; \sum_{-1\leq m\leq 1} e^{Nmh}\ \mathcal{Z}_m(\beta) \tag{16}$$

where

$$\mathcal{Z}_m(\beta) = \sum_{i=1}^{g_m} \quad e^{-\beta U_i} \tag{17}$$

is the partition function of a sample of REM with $g_m$ instead of $2^N$ states. If we write $g_m = (q_m)^N$, then all the results of last section concerning $\mathcal{Z}$ are valid for $\mathcal{Z}_m$ with the sole condition of replacing $\log 2$ by

$$\log q_m = \log 2 - \frac{1}{2}\left[(1+m)\log(1+m) + (1-m)\log(1-m)\right]$$

(for large $N$).

These restricted partition functions $\mathcal{Z}_m$ can be written as

$$\mathcal{Z}_m(\beta) = \sum_{-\infty<u<\infty} n(u,m)\; e^{-N\beta u} \tag{18}$$

where $n(u,m)$ are the numbers of states with internal energy $Nu$ and magnetization $Nm$. They can be seen to satisfy

$$\begin{cases} \langle n(u,m)\rangle \;\sim\; \exp N(\log q_m - u^2) \\ \langle n(u,m)^2\rangle - \langle n(u,m)\rangle^2 \;\sim\; \langle n(u,m)\rangle \end{cases} \tag{19}$$

Then for a typical sample we will have

$$\begin{cases} n^{\rm typ}(u,m) \sim \langle n(u,m)\rangle + \eta(u,m)\langle n(u,m)\rangle^{1/2} & \text{,if} \quad |u| < u_c(m) \\ n^{\rm typ}(u,m) \sim 0 & \text{,if} \quad |u| > u_c(m) \end{cases} \tag{20}$$

where the $\eta(u,m)$ are random numbers describing the typical fluctuations, and the energy threshold $u_c(m)$ is now $m$-dependent.

$$u_c(m) = (\log q_m)^{1/2} \tag{21}$$

The partition function of a typical sample under field will then be

$$\mathcal{Z}^{\rm typ}(\beta,h) = \sum_m e^{Nmh}\mathcal{Z}_m^{\rm typ}(\beta) \tag{22}$$

We now use the results of last section and write

$$\mathcal{Z}_m^{\rm typ}(\beta) = A_m(\beta) + B_m(\beta) \tag{23}$$

with

$$\begin{cases} A_m = \rho_m \exp\{N\beta u_c(m)\} & \text{, if} \quad \beta > 2u_c(m) \\ A_m = \rho_m \exp\{N\beta u_c(m)\} + \exp\{N(\log q_m + (\beta/2)^2)\} & \text{, if} \quad \beta < 2u_c(m) \end{cases} \tag{24}$$

and

$$\begin{cases} B_m = \eta_m \exp\{N\beta u_c(m)\} & \text{,if} \quad \beta > u_c(m) \\ B_m = \eta_m \exp\{\frac{N}{2}(\log q_m + \beta^2)\} & \text{,if} \quad \beta < u_c(m) \end{cases} \tag{25}$$

Here $\eta_m$ is a random number of order one which describes the fluctuations in the density of states as already mentioned. The origin of $\rho_m$ is somewhat different. They are due to fluctuations in the values of the lowest lying states: The ground state energy (internal energy in this case) $U_0$ has order one sample to sample fluctuations [12,13] so $U_0^{\rm typ}(m) \sim N(u_c(m) + \zeta/N)$ with $\zeta$ an order one random number so $e^{\beta U_0} \sim \rho e^{N\beta u_c(m)}$, where $\rho$ is a positive fluctuating number.

We see in this way that both in $A$ and $B$ the ground state contributions are non-analytic (due to disorder), and therefore non-integrable by the steepest descent method. The only analytic contribution is the saddle-point term in (24).

Let us now turn to the evaluation of the sum in (22). We define $\beta_c(m)$ as

$$\beta_c(m) = 2u_c(m) \tag{26}$$

and its inverse $m_c(\beta)$ such that satisfies

$$\beta/2 = \left[\log 2 - \frac{1}{2}\left[(1+mc)\log(1+mc) + (1-mc)\log(1-mc)\right]\right]^{1/2} \tag{27}$$

The relation $h_c(\beta) = \text{atanh}(m_c(\beta))$ defines the critical field $h_c$ which is the limit between the frozen and paramagnetic phases of REM with field [12]. This critical field satisfies

$$\beta/2 = [\log 2 + \log(\cosh(h_c)) - h_c \tanh(h_c)]^{1/2} \tag{28}$$

Using (26) we can rewrite the condition $\beta > 2u_c(m)$ as $|m| > m_c(\beta)$ and the condition $\beta > u_c(m)$ as $|m| > m_c(2\beta)$. The sum in (22) can now be written

$$\mathcal{Z}^{\text{typ}} = A + B \tag{29}$$

with

$$A = \sum_m A_m(\beta) e^{Nmh} \tag{30}$$

and

$$B = \sum_m B_m(\beta) e^{Nmh} \tag{31}$$

We then have

$$A = A_1 + A_2 = \sum_{-1 \le m \le 1} \rho_m e^{N[\beta u_c(m) + mh]} + \sum_{|m| < m_c(\beta)} e^{N[(\beta/2)^2 + \log q_m + mh]} \tag{32}$$

$$B = B_1 + B_2 = \sum_{|m| > m_c(2\beta)} \eta_m e^{N[\beta u_c(m) + mh]} + \sum_{|m| < m_c(2\beta)} \eta_m e^{\frac{N}{2}[\beta^2 + \log q_m + 2mh]} \tag{33}$$

The only analytic contribution is $A_2$, which will be estimated by means of the steepest descent method. We will ignore $B_1$ because it is already included in $A_1$. Let us first discuss the evaluation of the fluctuating contributions $A_1$ and $B_2$. The non-analyticity of the integrand makes it impossible to use the steepest descent method. On the other hand, as already discussed in section II, the term with the greatest modulus will dominate the sums so we have

$$|A_1| \sim \max_{-1 < m < 1} \quad \exp\{N[\beta(\log q_m)^{1/2} + mh_1]\} \tag{34}$$

and

$$|B_2| \sim \max_{|m| < m_c(2\beta)} \quad \exp\{\frac{N}{2}[\beta^2(\log q_m) + 2mh_1]\} \tag{35}$$

It is worth noticing that

$$|B_2| = 0 \qquad \text{if } \beta > \beta_c/2 \tag{36}$$

On the other hand, if $\beta < \beta_c/2$,then $m_c(2\beta) > 0$ and there exist two ranges of $h_1$ in which $B_2$ gives different contributions:

The maximum of $|B_2|$ is at $\bar{m} = \tanh(2h_1)$ when $|\bar{m}| < m_c(2\beta)$ and at $\pm m_c(2\beta)$ when $|\bar{m}| > m_c(2\beta)$. This can be written, using (27) and (28) as

$$|B_2| \sim \begin{cases} \exp\{\frac{N}{2}[\beta^2 + \log 2 + \log(\cosh(2h_1))]\} & ,\text{if} \quad |h_1| < \frac{1}{2}h_c(2\beta) \\ \exp\{N[\beta^2 + |h_1| m_c(2\beta)]\} & ,\text{if} \quad |h_1| > \frac{1}{2}h_c(2\beta) \end{cases} \tag{37}$$

Now we discuss the evaluation of $A_1$. The maximization condition is satisfied at $\hat{m} = \tanh(\hat{h})$ where $\hat{h}$ is a function of $h_1$ and $\beta$ that satisfies

$$\frac{\beta}{2}\frac{\hat{h}}{h_1} = \left[\log 2 + \log\cosh(\hat{h}) - \hat{h}\tanh(\hat{h})\right]^{1/2} \tag{38}$$

If we put $\hat{h} = h_c(\hat{\beta})$ with $\hat{\beta} = a\beta$ , now the unknown is $a(\beta, h_1)$. Replacing this in (38) we find $\hat{h} = ah_1$, so the following set of equations:

$$\begin{cases} \hat{\beta}/2 = [\log 2 + \log\cosh(\hat{h}) - \hat{h}\tanh(\hat{h})]^{1/2} \\ \hat{\beta}h = \beta\hat{h} \end{cases} \tag{39}$$

is equivalent to (38).

It is easy to see that when $h_1 = h_c(\beta)$ we obtain $a = 1$ while $a < 1$ if $h_1 > h_c(\beta)$ and $a > 1$ if $h_1 < h_c(\beta)$, so $\hat{h}$ will always be between $h_1$ and $h_c(\beta)$.
The term $A_1$ will then have the following form

$$|A_1| \sim \exp\{N[\frac{\hat{\beta}\beta}{2} + h_1\tanh(\hat{h})]\} \tag{40}$$

The calculation of $A_2$ may be done by the steepest descent method (see the appendix for details) for large $N$. It follows that again two different expressions are found, now depending on the values of $h$ and $\beta$. A smooth arc is found in the complex $h$ plane which divides these two possibilities. This arc touches the real axis at $h_c(\beta)$ and the imaginary axis at $h = i\pi/2$. In the inner part of this arc we find

$$A_2 \sim \exp N\{\frac{\beta^2}{2} + hm_c(\beta)\} + \exp N\{\frac{\beta^2 + \beta_c^2}{4} + \log(\cosh(h))\} \tag{41}$$

and in the outer side

$$A_2 \sim \exp N\{\frac{\beta^2}{2} + hm_c(\beta)\} \tag{42}$$

It is clear that $\mathcal{Z}$ will in general be the sum of several terms of the form $e^{N\varphi}$ with $\varphi$ a complex function, so for large $N$ just that term with the greatest modulus will be relevant. We then have to determine, in each region of the complex $h$ plane, the term with the greatest real part of $\varphi$. This will give rise to the appearance of a phase diagram for the model in complex h.

In what follows we will be only interested in the determination of the modulus of $\mathcal{Z}$, which, as we shall see, is enough to calculate the density of zeros of the model. If we write $|\mathcal{Z}| = \exp N\phi$, the different contributions we have to compare are

$$\begin{aligned}
\phi_1 &= \frac{\beta^2+\beta_c^2}{4} + \log|\cosh(h)| && \text{for} \quad \beta < \beta_c \quad \text{and} \quad |h_1| < h_1^{\lim}(h_2,\beta) \\
\phi_2 &= \frac{\beta^2}{2} + |h_1| m_c(\beta) && \text{for} \quad \beta < \beta_c \quad \text{and} \quad \forall h_1 \\
\phi_3 &= \frac{\beta\hat{\beta}}{2} + h_1 \tanh(\hat{h}) && \forall \beta \quad \text{and} \quad \forall h_1 \\
\phi_4 &= \frac{\beta^2}{2} + \frac{\beta_c^2}{8} + \frac{1}{2}\log(\cosh(2h_1)) && \text{for} \quad \beta < \frac{\beta_c}{2} \quad \text{and} \quad |h_1| < \frac{1}{2}h_c(2\beta) \\
\phi_5 &= \beta^2 + |h_1| \tanh(h_c(2\beta)) && \text{for} \quad \beta < \frac{\beta_c}{2} \quad \text{and} \quad |h_1| > \frac{1}{2}h_c(2\beta)
\end{aligned}$$

Three different ranges of temperature can be identified:

a) If $\beta > \beta_c$ then only $\phi_3$ exists so $\frac{\log|\mathcal{Z}|}{N} = \phi_3 \quad, \forall h$.

b) If $\beta_c > \beta > \beta_c/2$ we get contributions from $\phi_1, \phi_2$ and $\phi_3$. In this case two different regions are found in the complex $h$ plane (figure 1, upper portion), according to what term is the most important. These regions are separated by an arc which goes from $(h_c(\beta), 0)$ on the real axis to $(0, h_2^*(\beta))$ on the imaginary axis. In the inside of this arc $\phi_1$ dominates while in the outside it is $\phi_3$ which is the most important. $\phi_2$ is never relevant.

c) If $\beta < \beta_c/2$ then also $\phi_4$ and $\phi_5$ contribute, and they have to be compared to $\phi_1$ and $\phi_3$ in their respective zones. It is found that $\phi_5$ is never relevant, i.e. $(\phi_1 - \phi_5)$ and $(\phi_3 - \phi_5)$ are never negative. On the other hand $\phi_4$ is greater than $\phi_3$ (for $h_1 < \frac{1}{2}h_c(2\beta)$ ) and it is also greater than $\phi_1$ in the upper portion of the zone where $\phi_1 > \phi_3$, so the line where $\phi_4$ equals $\phi_1$ is lower than that where $\phi_3$ equals $\phi_1$.

The situation in case c) is that depicted in the lower portion of figure 1. The arc separating $\phi_1$ and $\phi_4$ now touches the imaginary axis at a point $(0, h_2^\dagger(\beta))$ which results from equating $\phi_4$ to $\phi_1$ for $h_1 = 0$ and satisfies

$$\cos(h_2^\dagger(\beta)) = e^{-(\frac{\beta_c^2 - 2\beta^2}{8})} \tag{43}$$

so we can see that $h_2^\dagger(\beta = 0) = \pi/4$.

## 4. DENSITY OF COMPLEX FIELD ZEROS OF REM

Having obtained $\phi = \frac{\log|\mathcal{Z}|}{N}$ in the various phases of the model, we can calculate the density of zeros $\rho(h)$ of $\mathcal{Z}$ by means of the formula [13]

$$\rho(h) = \frac{1}{2\pi}\nabla^2\phi = \frac{1}{2\pi}\left(\frac{\partial^2}{\partial h_1^2} + \frac{\partial^2}{\partial h_2^2}\right)\phi(h_1, h_2) \tag{44}$$

which is a consequence of the electrostatic analogy [2] that identifies $\rho(h)$ with a charge density and $\phi$ with the resulting electrostatic potential.

The density of zeros associated to $\phi_1$ is null (because the real part of an analytic function has a vanishing Laplacian). The densities of zeros corresponding to $\phi_3$ and $\phi_4$ are respectively

$$\rho_3 = \frac{1}{2\pi}\nabla^2\phi_3 = a(1 - \tanh^2(ah_1))\frac{\beta^2}{\beta^2 + 2h_1^2(1-\tanh^2(ah_1))} \tag{45}$$

$$\rho_4 = \frac{1}{2\pi}\nabla^2\phi_4 = 2(1 - \tanh^2(2h_1)) \tag{46}$$

where $a = \hat{h}/h_1$ as already defined.

In addition to these distributed zeros, there exist other zeros located right on the lines dividing the different zones, and which density is proportional to the discontinuity in the normal component of the "electric field" across that line. On a line separating two generic phases $a$ and $b$ we then have

$$\rho_l^{ab} = \frac{1}{2\pi} \mid \bar{\nabla}(\phi_a - \phi_b) \mid \tag{47}$$

The density of zeros on line 1-3 is found to be

$$\rho_l^{13} = \frac{1}{2\pi}\{(\tanh(\hat{h}) - \frac{\sinh(2h_1)}{\cosh(2h_1) + \cos(2h2)})^2 + (\frac{\sinh(2h_1)}{\cosh(2h_1) + \cos(2h2)})^2\} \tag{48}$$

On line 1-4 the expression is the same with the only difference that $\hat{h}(\hat{h} = ah_1)$ is replaced by $2h_1$. Their zero-densities are equal where lines 1-3 and 1-4 meet.

The line 3-4, which is the vertical at $h_1 = \frac{1}{2}h_c(2\beta)$ is not a line of zeros, i.e. the electric field is found to be continuous there. On the other hand the density of *distributed* zeros jumps as one crosses line 3-4 by a factor $\{1 + 2(h_1/\beta)^2(1 - \tanh^2(2h_1))\}$.

The density of zeros on the line 1-3 is found to be null at the point where that line touches the real $h$ axis, in agreement with the second order character of the field-driven transition of the model.

## 5. **CONCLUSIONS**

We have estimated the behavior of the partition function of REM in complex field. Three different phases appear:

For $T < T_c$ there is only one phase which we called phase 3 and is the continuation to complex field of the frozen phase of the model. For $T_c < T$ the paramagnetic phase is also present, which we call phase 1 (figure 1, upper portion). Finally for $T > 2T_c$ what we called phase 4 also appears at complex values of $h$ (figure 1, lower portion). There is no physical (i.e. for *real* values of $h$) counterpart for this phase.

We have also calculated the densities of zeros in the different phases. A particular structure has been revealed: The zeros are dense near the real axis in the frozen phase. This characteristic of the zeros distributions is intimately related to the properties of spin glasses and has been discussed by Parisi [11], although no explicit calculation had been done before. A numerical estimate of the complex field zeros of REM has been done [17] and shows good coincidence [18] with the results here reported.

The existence of distributed zeros is in this model related to disorder in a very clear way: The fluctuating factors $\rho_m$ and $\eta_m$ are responsible for the appearance, in $\mathcal{Z}$, of nonanalyticities of the sort which are necessary to obtain a non-zero Laplacian and consequently distributed zeros.

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## APPENDIX

Here we describe the evaluation of $A_2$ by means of the steepest descent method [16]. Starting from

$$A_2 = \int_{-m_c(\beta)}^{m_c(\beta)} \mathrm{d}m \; e^{N\Psi(m)} \tag{A1}$$

with

$$\Psi(m) = mh + \log 2 - \frac{1}{2}\left[(1+m)\log(1+m) + (1-m)\log(1-m)\right] + \left(\frac{\beta}{2}\right)^2 \tag{A2}$$

we see that $A_2$ is non-zero only for $\beta < \beta_c$, otherwise $m_c(\beta) = 0$ and this term does not contribute.

Following the usual procedure for evaluating this kind of integrals in the complex plane, we stretch the integration contour to infinity following paths of steepest descent for the real part of $\Psi$ . The imaginary part of the integrand will in this way be constant along the whole path, and may be taken out from the integral. The level curves of $\Psi$ are topologically similar to those of $z^2$. Two steepest descent lines cross each other at the saddle point which is located at $m_0 = \tanh(h)$: one of them goes down to the valleys to the left and right of the saddle-point and will be called AA' while the other climbs towards the mountain tops and is the line LL'. The way in which the original integration path has to be deformed depends on the relative positions of $\pm m_c$ and LL'.

Depending on $\beta$ and $h$, two situations are possible:

a) When the integration limits $\pm m_c(\beta)$ lay on opposite sides of LL', there are contributions from three paths: the one that runs through the saddle-point (path AA'), and those ending at $\pm m_c(\beta)$.

b) When both integration limits are on the same side of LL', the path that joins the valleys through the saddle-point does not contribute to the integral.

The limit condition for this two cases is that $m_c(\beta)$ lays exactly on LL', which in turn is equivalent to the condition that $\mathrm{Imag}(\Psi(m_c))$ $=\mathrm{Imag}(\Psi(m_0))$. This last identity results from the fact that LL', being a steepest descent for the *real* part of $\Psi$, is a level curve for the *imaginary* part of $\Psi$. This limit condition is then written

$$\tanh(h_1) = \frac{\tan(m_c(\beta)h_2)}{tg(h_2)} = \tanh(h_1^{\lim}) \tag{A3}$$

and for $\beta < \beta_c$ it defines a smooth arc in the complex $h$ plane (with $h_1$ and $h_2$ positive) that joins the points $(h_c(\beta), 0)$ and $(0, \pi/2)$. The shape and size of this arc depend on $\beta$. In its inner part situation a) is valid so for large $N$

$$A_2 \sim \exp N\{\frac{\beta^2}{2} + hm_c(\beta)\} + \exp N\{\frac{\beta^2 + \beta_c^2}{4} + \log(\cosh(h))\} \tag{A4}$$

while in the outer side of this arc b) holds and

$$A_2 \sim \exp N\{\frac{\beta^2}{2} + hm_c(\beta)\} \tag{A5}$$

**REFERENCES**

[1] C. N. Yang and T. D. Lee Phys. Rev. **87**(1952),404

[2] C. N. Yang and T. D. Lee Phys. Rev. **87**(1952),410

[3] Fisher M. in *Lectures in Theoretical Physics* **7C** (University of Colorado Press, Boulder, Colorado, 1965)

[4] G. Jones J. Math. Phys. **7**(1966),2000

[5] van Saarloos W. and Kurtze D J. Phys. A :Math. Gen. **17**,1301(1984)
J. Stephenson and R. Couzens Physica A **129** (1984),201

[6] Suzuki M. Prog. Theor. Phys. **38** (1967), 1243, 1255
Ono S. et al J. Phys. Soc. Japan **26s** (1969), 96
Abe R. Prog. Theor. Phys. **38** (1967), 72 , 322

[7] C. Itzykson et al Nuc. Phys. **B220**[FS8] (1983), 415

[8] M. L. Glasser et al. Phys. Rev. B **35** (1987), 1841

[9] G. Bhanot et al. Phys. Rev. Lett. **59** (1987), 803
E. Marinari Nuc. Phys. **B235** [FS11] (1984), 123

N. Alves et al. Phys. Rev. Lett. **64** (1990), 3107

[10] *Spin Glasses and Beyond.* Mezard M. , Parisi G. and Virasoro M.
World Scientific. Lecture Notes in Phys. Vol 9

[11] G. Parisi, in *Disordered systems and Localization*
Springer Lecture Notes in Phys. Vol. **149**.

[12] B. Derrida Phys. Rev. B **24** (1981), 2613
B. Derrida, Phys. Rep. **67** (1980), 29.

[13] B. Derrida (1991) Physica A **177** (1991), 31.

[14] C.Moukarzel and N.Parga, Proceedings of the *Second Latin American Workshop on Nonlinear Phenomena* , Santiago de Chile, September 1990 (North Holland 1991).

[15] C.Moukarzel and N.Parga , Physica A **177** (1991), 24

[16] *Asymptotic Expansions* E. T. Copson (Cambridge Univ. Press, 1971)

[17] C.Moukarzel and N.Parga (1991), unpublished.

[18] C.Moukarzel and N.Parga (1991) *The REM Zeros in the Complex Temperature and Magnetic Field Planes.*, Physica A **185** (1992), 305.

## FIGURE CAPTIONS

FIGURE 1. Shown are the phases of the model in the complex field plane for two different values of the temperature. For $T_c < T < 2T_c$(upper figure) two distinct zones appear separated by an arc of zeros. Above this arc (phase 3) densely distributed zeros are found while the region below it (phase 1) is free of zeros. For $T > 2T_c$ the outer part of the arc appears divided in two zones by a vertical line, which is not a line of zeros. The densities of zeros are different to the left (phase 4) and right (phase 3) of this vertical line.

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